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Rotating Frames in Special Relativity

M. STRAUSS

Frankfurter Tor 1, 1034 *Berlin, GDR*

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Abstract

A uniformly rotating frame is defined as the rest frame of a particle revolving with constant velocity ω in a circle about the Z-axis of an inertial frame Σ^0. Under the condition $z = Z$, $r = R$, theoretical constraints are established for the solution of the transformation problem $\Sigma^0 \to \Sigma^{\omega}{}_r$, $\Sigma^{\omega}{}_r$ being the cylindrical subframe of Σ^{ω}. The unique solution of the problem in cylindrical coordinates is isomorphic to the special Lorentz transformation L_x, with $\beta = v/c$ replaced by $\beta_r = \omega r/c$. Hence the intrinsic geometry on the surface of a rotating cylinder is Euclidean. Though there exists no complete intrinsic geometry on the surface of a rotating disk, the geodesics on it are straight lines while the circumference of a concentric circle is $K_r 2\pi r$ as predicted by Einstein.

1. *Introduction*

Since Einstein (1917, 1921) suggested that the physical geometry on a rotating disk would turn out to be non-Euclidean, rotating frames were usually considered to fall outside the domain of Special Relativity. The mistakes involved in this conclusion are as follows. First, the domain of Special Relativity is given by the condition $R_{iklm} = 0$ (absence of a true gravitational field) which yields Minkowski space $M = E_{3+1}$. A three- or two-dimensional subspace of M will of course in general have a non-Euclidean intrinsic geometry, just as a surface in E_3 will in general be curved and not flat. Thus, even if we could accept Einstein's argument for non-Euclidicity as correct, it would not follow that rotating frames fall outside the scope of Special Relativity. Second, Einstein's argument, in my view, is wrong: if the measuring rods laid along the circumference of the rotating disk are Lorentz contracted with respect to the inertial frame, so are the distances on the circumference they are supposed to measure; hence the two effects would cancel each other and the ratio C/D (circumference/diameter) would turn out to equal π as in the Euclidean plane. Thus, Einstein's thought experiment, far from establishing non-Euclidicity for the intrinsic geometry on the rotating disk, suggests in fact that this geometry be Euclidean.

Incidentally, Einstein used this thought experiment to conclude that *in the presence of a gravitational field* physical geometry would be non-Euclidean. Though the conclusion happened to be right, the argument, in my view, is wrong for two reasons. The first has just been pointed out above. The second thing I consider mistaken is the interpretation of the field of *inertial forces* (centrifugal and Coriolis) as a true *gravitational* field. To be sure, inertial forces *can* be interpreted as gravitational forces (due to distant masses) as suggested by Mach, but (so far) only within Newtonian mechanics (Strauss, 1968, 1972) or a non-Newtonain mechanics with Newtonian space-time (Treder, 1972a, b), and certainly not in Einstein's theory of gravitation: the 'Principle of Equivalence' holds only locally, and even there only in the approximation where the derivatives of the field can be neglected.

To say that rotating frames, in the absence of gravitational fields, do belong to the domain of Special Relativity does not mean that General Relativity is inapplicable or should not be used. Yet so far it has not been used correctly. The mistake, repeated over and over again, lies in the method used, viz., the use of a Galilean transformation for the introduction of coordinates in the rotating frame. This has two consequences apparently not realised by the authors. First, the spatial coordinates so introduced have no metrical significance (what the authors know of course) so that the geodesics equations—which are equations in these coordinates—have no physical significance either (what the authors fail to realise). Second, the time coordinate t introduced by the Galilean transformation ($t = T$) is a *frame time* (the same for all points in the frame), while a correct treatment of rotating frames must use *local times* $t^{(r)}$ depending on the radius r because standard clocks revolving in different circles and hence with different speeds cannot be synchronised to show the same time all the time. Hence all consequences involving t = const. (such as the derived 'geometry on the rotating disk') are incorrect. In fact, the method described gives not only wrong results but leads to inconsistences: when applied to the transformation between inertial frames in M it yields non-Euclidean geometries for all inertial frames except the original one (see Appendix). Since every inertial frame can be taken as the original one, the results of this method are apparently self-contradictory.

The key problem in the relativistic theory of rotating frames is that of finding the correct transformation formulae for the transition $\Sigma^0 \to \Sigma^{(r)}{}_\omega$ where Σ^0 is an inertial *kinematic* (space-time) frame and where the second symbol means the local kinematic frame at any point P revolving with respect to Σ^0 in a circle with radius $r = R$ and constant angular velocity ω: all other problems in that theory depend on the solution of this problem. Now this key problem is a genuine problem (in the sense of problem theory) since no algorithm for its solution is known (we may even safely assume that no such algorithm exists): this explains why different solutions are offered, none of which has been generally accepted. In other words, a relativistic theory of rotating frames is a genuine *extension* of standard Special Relativity involving, more likely than not, additional postulates and/or definitions. (We shall only use a definition of *non-inertial standard time* on which all authors are agreed.)

For attacking a genuine problem there exist various strategies, from a more or less inspired guessing to a strategy of theoretical restraints. The guessing strategy is essentially a 'trial and error' strategy but it may involve any kind of heuristic arguments. In the case of our problem the heuristic argument is an obvious analogy—the group parameter $\beta = v/c$ of the special Lorentz group corresponds to $\beta^{(r)} = \omega r/c$—and this suffices to make the guess unique. The theoretical constraints result in our case from the fact that some of the consequences of the correct solution are known or can be established without knowing the correct solution itself.

Section 2 reduces the key problem to its irreducible core, viz., to a two-dimensional transformation problem. Section 3 gives the theoretical constraints for its solution, which are shown to be satisfied by the surmised solution. Section 4 gives some results and their discussion, including those pertaining to the chrono-geometry and the propagation of light in a rotating frame.

2. *The Irreducible Core of the Problem*

Let $\Sigma^0 = \{Z, R, \theta, T\}$ be an inertial frame with the metrical time-orthogonal cylindrical coordinates Z, R, θ and the common frame time T as measured by synchronised standard clocks. Then:

$$dS^2 = dZ^2 + dR^2 + R^2\, d\theta^2 - c^2\, dT^2 \tag{2.1}$$

Consider a point A revolving about the Z-axis at fixed distance R_A with constant angular velocity $\omega_A{}^0$ with respect to Σ^0 so that

$$\text{(i)}\ d\theta_A/dT = \omega_A{}^0, \qquad \text{(ii)}\ dR_A/dT = dZ_A/dT = 0 \tag{2.2}$$

In the rest-frame of A, Σ^A, we introduce the coordinates $z, r, \vartheta^{(r)}, t^{(r)}$ with

$$\text{(i)}\ z = Z, \qquad \text{(ii)}\ r = R \tag{2.3}$$

$t^{(r)}$ is the local time variable at any point $r = R$ and $\vartheta^{(r)}$ the local angular variable at any point $r = R$. Then

$$dS^2 = dz^2 + dr^2 + r^2 \left[\frac{\partial\theta}{\partial\vartheta} d\vartheta + \frac{\partial\theta}{\partial t} dt\right]^2 - c^2 \left[\frac{\partial T}{\partial\vartheta} d\vartheta + \frac{\partial T}{\partial t} dt\right]^2 \tag{2.4}$$

where the upper index (r) has been suppressed. It should be noted that here and in the following the symbol 'r' plays a double role: if not used as an index, 'r' is a free (independent) coordinate variable with respect to which differentiation can take place; if used as a (lower or upper) index (with or without brackets) it serves as a parameter with respect to which no differentiation will take place. Thus $t^{(r)}$, when expressed as a function of T and θ, will depend parametrically on r, besides of course depending parametrically on the relative angular velocity $\omega^0{}_A$. To avoid confusion, and to unify the notation, the parameter r_A will be replaced by the (dimensionless) parameter

$$\beta^0{}_A =_{df} \omega^0{}_A r_A/c \tag{2.5}$$

where $\omega^0{}_A$ is now defined by

$$\omega^0{}_A = {}_{df}[d\theta/dT]_{dz=dr=d\vartheta^{(r_A)}=0} \tag{2.6}$$

By (2.6) it is ensured that the angular velocity with respect to Σ^0 is the same for all points at rest in Σ^A as befits the proper definition of '(constant) angular velocity' and 'rest-frame Σ^A'. We shall assume that the inverse angular velocity defined by

$$\omega_0{}^{A,r} = {}_{df}[d\vartheta^{(r)}/dt^{(r_A)}]_{dZ=dR=d\theta=0} \tag{2.7}$$

i.e. the angular velocity of any point resting in Σ^0 with respect to Σ^A is likewise the same for all reference points in Σ^A and hence independent of the parameter r so that the upper index 'r' on the left-hand side can be dropped:

$$\omega_0{}^{A,r_1} = \omega_0{}^{A,r_2} = \ldots = \omega_0{}^A \tag{2.8}$$

We may further assume that

$$\omega_0{}^A = -\omega_A{}^0 \tag{2.9}$$

Equations (2.8) and (2.9) can be considered as definitional restrictions on the transformation problem, but they do in fact follow from the theoretical restrictions given in the next section: their anticipation merely serves to simplify the notation.

By equations (2.2) the transformation problem is reduced to a two-dimensional problem, viz., to finding the mapping operator $\mathcal{R}^{0A}$ defined by

$$\begin{pmatrix}\theta\\ T\end{pmatrix} = \mathcal{R}^{0A}\begin{pmatrix}\vartheta^{(r)}\\ t^{(r)}\end{pmatrix} \tag{2.10}$$

or its inverse $\mathcal{R}^{A0}$ satisfying

$$\mathcal{R}^{0A}\mathcal{R}^{A0} = \mathcal{R}^{A0}\mathcal{R}^{0A} = I \qquad \text{(identity)} \tag{2.11}$$

If we write $\mathcal{R}^{0A}$ in the form

$$\mathcal{R}^{0A} = \mathcal{R}(\beta_A{}^0; \omega_A{}^0) \tag{2.12}$$

it follows that

$$\mathcal{R}^{A0} = \mathcal{R}(\beta_0{}^A; \omega_0{}^A) = \mathcal{R}(-\beta_A{}^0; -\omega_A{}^0) \tag{2.13}$$

Thus the problem is that of finding the mapping operator $\mathcal{R}$ as a function of the two relational quantities (parameters) $\omega_A{}^0$ and $\beta_A{}^0$, of which the second one involves also the radius r as parameter.

In view of the obvious analogy to the problem of special Lorentz transformation $\mathscr{L}_x$ (with $R\,d\theta$ corresponding to dX and $r\,d\vartheta^{(r)}$ corresponding to dX') we venture the guess

$$\begin{aligned} \vartheta^{(r)} &= \kappa_r[\theta - \omega T] \\ t^{(r)} &= \kappa_r[-(\beta_r{}^2/\omega)\theta + T] \end{aligned} \tag{2.14}$$

where

$$\kappa_r = [1 - \beta_r^2]^{-1/2} \tag{2.15}$$

which gives for the inverse transformation

$$\begin{aligned} \theta &= \kappa_r[\vartheta^{(r)} + \omega t^{(r)}] \\ T &= \kappa_r[(\beta_r^2/\omega)\vartheta^{(r)} + t^{(r)}] \end{aligned} \tag{2.16}$$

Applying the definitions (2.6) and (2.7) we identify the relational parameter ω:

$$\omega = \omega_A{}^0 = -\omega_0{}^A \tag{2.17}$$

Thus, the suggested solution of the irreducible core of our problem reads

$$\mathcal{R}_r^{0A} = \mathcal{R}(\beta_r{}^0; \omega_A{}^0) = \kappa_r \begin{bmatrix} 1 & \omega_A{}^0 \\ \beta_r^{02}/\omega_A{}^0 & 1 \end{bmatrix} \tag{2.18}$$

Note that the (matrix) operator valued function $\mathcal{R}$ suggested in (2.18) satisfies the symmetry condition (2.13) since $r = R$. Hence the relation between the coordinates $\{\theta, T\}$ and $\{\vartheta^{(r)}, t^{(r)}\}$ is completely symmetrical. However, while the former are *global* coordinates the latter are merely *cylindrically extended local* coordinates defining a (cylindrical) *subframe*. Hence the relation between the (global) inertial frame Σ^0 and the (total) rotating *frame* Σ^A is *not* symmetrical. (This is one of the theoretical constraints established in the next section.)–Note that (2.16) implies *form invariance* of the interval:

$$dZ^2 + dR^2 + R^2\,d\theta^2 - c^2\,dT^2 = dz^2 + dr^2 + r^2\,d\vartheta^{(r)2} - c^2\,dt^{(r)2} \tag{2.19}$$

3. *Theoretical Constraints*

C1. (*Non-relativistic limit.*) In the non-relativistic limit ($c \to \infty$) the transformation must go over into $\vartheta^{(r)} = \theta - \omega T$, $t^{(r)} = t = T$.

C2. (*Local time metric.*) The transformation must be such that the implied local time metric at any point in Σ^A moving with the speed $c\beta_r$ with respect to Σ^0 differs from the local time metric in Σ^0 by the factor κ_r^{-1}, i.e.,

$$\left[\frac{dt^{(r)}}{dT}\right]_{dr = d\vartheta^{(r)} = 0} = \kappa_r^{-1} \tag{C2}$$

Comment. Strictly speaking (C2) *defines* the local time metric in the rotating frame; clocks that realise this time metric may be called *standard clocks for the rotating frame*. However, the definition would be unpracticable if there were no such standard clocks. Hence the definition implies that at least some of the ordinary standard clocks are also standard clocks for the rotating system. The most obvious candidate for being such a standard clock is the radioactive one.

For the *implied geometry* in the rotating frame no theoretical constraint will be given as no such constraint seems to be firmly established.

The following constraints refer to the 1-parameter family of frames rotating with different angular velocities about the common Z-axis. The following notation will then be used. Different frames are distinguished by different capitals ($A, B, \ldots$), their angular velocities with respect to the inertial frame Σ^0 are denoted by $\omega_A, \omega_B, \ldots$, i.e., the upper index '0' is dropped. The angular velocity of Σ^B with respect to Σ^A will be denoted by $\omega_B{}^A$. The cylindrical subframe of Σ^A is characterised by its radius r_A. By definition

$$\beta^A_{BB} = \omega_B{}^A r_B/c, \qquad \kappa^A_{BB} = [1 - \beta^A_{BB}{}^2]^{-1/2} \tag{3.1}$$

Dropping the upper index implies again that the reference frame is the inertial frame Σ^0.

C3. (*General uniform rotation equivalence.*) The 2-parameter family of cylindrical subframes rotating with different angular velocities about a common axis in an *equivalence class in the strong sense,* i.e., the transformations between these frames form a group generated by a *single* operator valued function of the 2 x 2 parameters involved (in other words: the transformation operators are the *same function* of the four parameters involved):

$$\mathcal{R}^{AB} = \mathcal{R}(r_A, r_B; \omega_A; \omega_B) \qquad \text{for } all\ A, B \tag{C3}$$

Comments. The group property follows already from the fact that $\mathcal{R}^{AB} = \mathcal{R}^{A0}\mathcal{R}^{0B}$ and is therefore no constraint on $\mathcal{R}^{A0}$, apart from the trivial requirement that these operators must possess one and only one inverse. Since every transformation group defines an equivalence relation, viz., the relation holding between objects connected (mapped one to the other) by an element of the group, the requirement that the family of rotating frames or subframes is an equivalence class is likewise trivial and hence no constraint on the operators $\mathcal{R}^{0A}$. Thus the constraint lies in the *italicised* words. Note that (C3) is the weakest possible constraint that ensures that all frames or subframes uniformly rotating about a common axis are truly equivalent: a much stronger constraint would be the demand that $\mathcal{R}^{AB}$ is not a function of both ω_A and ω_B but merely of $\omega_B{}^A$. This demand would mean that any reference to the inertial frame Σ^0 should disappear from the general transformation equations and this demand can not be justified in view of the fact that the inertial frames play a distinguished role in Minkowski space M: there is only one uniform motion equivalence in M (while in Newtonian space-time $E_3 \times T$ there exists a transfinite set of uniform motion equivalences one of which is singled out by the equations of motion and called the class of inertial frames) which may therefore be identified with the Newtonian class of inertial frames. [Thus, in Special Relativity the term 'inertial frame' does not involve any reference to the equations of motion (dynamics) but only to the space-time structure (kinematics)]. Furthermore, the frames of the uniform motion equivalence in M are distinguished by the fact that they are *global* frames while all other frames are not. Thus, the rotating frames are composed of cylindrically extended local frames, which is the reason why the cylinder radius of the subframe appears as a parameter.

If all radii are equal ($r_A = r_B = \cdots = r$) we obtain a subgroup of the general rotational transformation group. The corresponding subfamily of rotating frames is essentially a one-parameter family, the parameter being the angular velocity $\omega = \omega^0$. (The additional parameter r, now being the same for all frames considered, is now merely an invariant constant, just like c.) Hence the transformation connecting any two of these frames can at most be a two-parameter transformation: $\mathcal{R}^{AB} = \mathcal{R}_2(\omega_A{}^0, \omega_B{}^0)$. The constraint we wish to impose is that it is in fact a one-parameter transformation: $\mathcal{R}^{AB} = \mathcal{R}_1(\omega_B{}^A)$. Though plausible enough, the constraint may be objected to on the ground that it eliminates any reference to the inertial frame Σ^0. The reply is as follows. There is no reason why we should exclude the cylindrical *subframe* $R = r$ of Σ^0 from the one-parameter subfamily considered (if we allow the parameter ω^0 to take the value 0 the cylindrical subframe $R = r$ of Σ^0 is automatically included in the family). If this is accepted, the index '0' is no longer distinguished (as it is in the arguments of $\mathcal{R}_2$) and we *must* have $\mathcal{R}^{AB} = \mathcal{R}_1(\omega_B{}^A)$ where the indices may now take the 'value' '0'. We have thus established the following constraint:

C4. (*Subgroup* $r_A = r_B = \ldots = r$.) (*Cylinder subgroup.*) The cylinder subgroup of the general rotational transformation group must turn out to be a one-parameter Lie group, i.e.,

$$\mathcal{R}(r, r; \omega_A, \omega_B) = \mathcal{R}_r(\omega_B{}^A) \tag{C4}$$

Comments. Note that the argument used to establish C4 cannot be used to reduce the number of parameters in the general rotation group characterised by (C3): if we put $\omega_A = 0$ the frame Σ^A becomes identical with the inertial frame Σ^0 so that there is only one rotating frame instead of two, i.e., we are no longer considering the general case.

The one-parameter kinematic transformation problem in two dimensions (one spatial, one temporal) has been studied by many authors, the present one included (Strauss, 1957/58, 1966). The result may be stated thus: if the frames considered are supposed to form an equivalence class in the strong sense as explained above, the transformation problem has exactly two solutions one of which is the special Lorentz transformation while the second solution results from the first one by equating a certain invariant (which is equated in the first solution to c^2) to $-C^2$. (C is then not a limiting but a critical velocity in a closed Euclidean space.) As the second solution is to be excluded for physical reasons only the first solution remains. Thus we have the further constraint:

C5. (*Cylinder subgroup.*) The cylinder subgroup of the general rotational transformation group, considered in C4, must turn out to be formally identical with the special Lorentz group:

$$\mathcal{R}_r(\omega) = \mathscr{L}(\beta_r), \qquad \beta_r = r\omega/c = \tau\omega \tag{C5}$$

where $\tau = r/c$ is the invariant time constant of this group.

Comment. This constraint is of course the strongest one as it determines the solution of our key problem completely and directly. However, it is unavoidable if the other constraints are accepted.

There is of course the further subgroup defined by the condition $\omega_A = \omega_B = \ldots = \omega$, which may be called the *rotating disk subgroup.* It will be studied in the next Section. No constraint referring to it need or can be given, as the solution of our problem is already completely determined.

4. *Results and Discussion*

(a) *The general uniform rotation equivalence*

As a consequence of (2.18) we have

$$\mathcal{R}^{A\,B}_{r_A r_B} = \mathcal{R}^{A0}_{r_A} \cdot \mathcal{R}^{0B}_{\;\;r_B} = \kappa_{AA}\kappa_{BB} \left[\begin{array}{c|c} 1 - \beta^2_{BB}\dfrac{\omega_A}{\omega_B} & \omega_B - \omega_A \\ \hline \dfrac{\beta^2_{BB}}{\omega_B} - \dfrac{\beta^2_{AA}}{\omega_A} & 1 - \beta^2_{AA}\dfrac{\omega_B}{\omega_A} \end{array}\right] \tag{4.1}$$

This is the matrix operator connecting two arbitrary elements (cylindrical subframes) of the general uniform rotation equivalence. The constraint (C3) is immediately verified, having regard to the definitions (3.1). The group properties

$$\mathcal{R}^{AA}_{rr} = I, \qquad \mathcal{R}^{BA}_{r_B r_A} = (\mathcal{R}^{AB}_{r_A r_B})^{-1}$$

$$\mathcal{R}^{AC}_{r_A r_C} = \mathcal{R}^{AB}_{r_A r_B} \cdot \mathcal{R}^{BC}_{r_B r_C} \tag{4.2}$$

which follow from the product construction may also be directly verified. The relative angular velocity of Σ^A with respect to subframe $\Sigma^B_{r_B}$, defined by

$$\omega_A{}^B =_{df} [d\vartheta^{(r_B)}/dt^{(r_B)}]_{d\vartheta^{(r_A)}=0} \tag{4.3}$$

works out to

$$\omega_A{}^B = \frac{\omega_B - \omega_A}{1 - \omega_A\,\omega_B r_B{}^2 c^{-2}} \tag{4.4}$$

Hence

$$\omega_A{}^B \neq -\omega_B{}^A \qquad \text{unless } r_A = r_B \tag{4.5}$$

This apparent lack of symmetry is due to the fact that the time metric depends on β_r and hence on both the angular velocity and the radius.

(*b*) *The rotating cylinder subgroup* ($r_A = r_B = \ldots = r$)

If all cylindrical subframes have the same radius r the general transformation operator (4.1) reduces to

$$\mathcal{R}^{AB}_{r\,r} = \kappa_{AB}\left(\begin{array}{c|c} 1 & \omega_{AB} \\ \hline \omega_{AB}(r^2/c^2) & 1 \end{array}\right) \tag{4.6}$$

where κ_{AB} is defined by

$$\kappa_{AB} =_{df} [1 - (\beta_B{}^A)^2]^{-1/2} \tag{4.7}$$

and satisfies the mathematical identity (composition law)

$$\kappa_{AB} = \kappa_A \kappa_B [1 - \beta_A \beta_B] \tag{4.8}$$

which has been used in obtaining (4.6).

Thus, the rotating cylinder subgroup is isomorphic to the special Lorentz group as required by C5.

(c) *The rotating disk subgroup* ($\omega_A = \omega_B = \cdots = \omega$)

If all cylindrical subframes have the same angular velocity ω with respect to Σ^0 they define a single frame (massive cylinder) with r-depending time metric. Different values of r, referring to the different subframes, will be distinguished by different numerical indices. This frame may be bodily represented be a rotating disk, the different subframes by different concentric circles. Alternatively, we may think of particles revolving on concentric circles with the same angular velocity.

The transformation operator linking two subframes now reduces to

$$\mathcal{R}^{AA}_{r_1 r_2} = \kappa_1\kappa_2\left(\begin{array}{c|c} 1-\beta_2{}^2 & 0 \\ \hline \dfrac{\beta_2{}^2-\beta_1{}^2}{\omega} & 1-\beta_1{}^2 \end{array}\right) = \left(\begin{array}{c|c} \kappa_1\kappa_2^{-1} & 0 \\ \hline \dfrac{\kappa_1^{-1}\kappa_2-\kappa_1\kappa_2^{-1}}{\omega} & \kappa_1^{-1}\kappa_2 \end{array}\right) \tag{4.29}$$

Hence

$$\left.\begin{aligned} \vartheta^{(1)} &= \kappa_1\kappa_2^{-1}\vartheta^{(2)} \\ t^{(1)} &= (\kappa_1^{-1}\kappa_2 - \kappa_1\kappa_2^{-1})\omega^{-1}\vartheta^{(2)} + \kappa_1^{-1}\kappa_2 t^{(2)} \end{aligned}\right\} \tag{4.10}$$

Rearranging, we see that the following two quantities are invariant under the subgroup considered, i.e., independent of the radius:

$$\kappa_1^{-1}\vartheta^{(1)} = \kappa_2^{-1}\vartheta^{(2)} \equiv \bar{\vartheta} \tag{4.11}$$

$$\kappa_1 t^{(1)} + \kappa_1\omega^{-1}\vartheta^{(1)} = \kappa_2 t^{(2)} + \kappa_2\omega^{-1}\vartheta^{(2)} \equiv I_t \tag{4.12}$$

While (4.11) expresses the Lorentz contraction, the significance of (4.12) will be explored in the next subsection.

(d) *Digression: Frame coordinates for the rotating disk*

Equation (4.12) permits the introduction of a frame time $\bar{t}$, the same for all subframes. Having regard to the physical dimensions, the most general equation introducing $\bar{t}$ is

$$I_t = \bar{t} f(\omega \bar{t}, \bar{\vartheta}) \tag{4.13}$$

where f is an arbitrary function. Thus, while an r-independent angular measure is uniquely fixed by (4.11), an r-independent time measure is not uniquely determined by the subgroup characterising a rotating disk.

One arbitrary but almost self-suggesting way to determine the function f is as follows. Consider the angular quantity

$$\hat{\theta} =_{df} \omega \bar{t} f(\omega \bar{t}, \bar{\vartheta}) \tag{4.14}$$

which, on account of the factor ω, obviously refers to the inertial frame Σ^0. Then:

$$\bar{\omega} =_{df} \left[\frac{d\bar{\vartheta}}{dt}\right]_{d\hat{\theta}=0} = -\frac{f + y f_y}{y f_x} \omega \qquad (x = \bar{\vartheta}, y = \omega \bar{t}) \tag{4.15}$$

where the lower indices mean partial differentiation. If we identify $\bar{\omega}$ with $-\omega$ (and hence $\hat{\theta}$ with θ) the resulting partial differential equation for f has the general solution

$$f = a[1 + x/y] = a[1 + \bar{\vartheta}/\omega \bar{t}] \tag{4.16}$$

where a is an arbitrary numerical constant. This yields

$$\theta = a[\bar{\vartheta} + \omega \bar{t}] \tag{4.17}$$

i.e. (apart from the numerical constant a) exactly the Galilei transformation. Hence, if we use the function (4.16), the frame time $\bar{t}$ introduced by (4.15) has to be identified with the Newtonian time, and vice versa. This, to be sure, does not constitute a justification of the use of the Galilei transformation: not only is the choice of the function (4.16) completely arbitrary from the point of view of Special Relativity as outlined above, but the very concept of a frame time common to all points of a rotating frame is at best a mathematical construct without proper physical meaning.

(e) *Chrono-geometry in the rotating frame*

From equations (4.10) we obtain

$$\kappa_1^{-1} \Delta\vartheta^{(1)} = \kappa_2^{-1} \Delta\vartheta^{(2)} \tag{4.18}$$

$$\kappa_1 \Delta t^{(1)} = \kappa_2 \Delta t^{(2)} \qquad \text{iff } \Delta\vartheta^{(1)} = \Delta\vartheta^{(2)} = 0 \tag{4.19}$$

These relations correspond to the well-known Lorentz contraction and time dilation of which they are a direct consequence as may be seen from the fact that they are derivable from equations (2.14) and (2.16), respectively. However there is a significant difference: while the latter hold between different frames, equations (4.18), (4.19) hold between different subframes of the *same frame*. Hence they express the *intrinsic* chrono-geometry of the rotating frame. Moreover, while the Lorentz contraction is a projection effect $[\Delta t = 0]$, equation (4.18) holds independently of any condition involving simultaneity and indeed independently of any condition whatsoever. In other words, (4.18) is not a chrono-geometrical but a *purely geometrical relation*, in contrast to (4.19) which is not a purely chronometrical relation.

Indeed, if the condition $\Delta\vartheta^{(1)} = \Delta\vartheta^{(2)} = 0$ is not satisfied we merely obtain

$$\kappa_1 [\Delta t^{(1)} + \omega^{-1} \Delta\vartheta^{(1)}] = \kappa_2 [\Delta t^{(2)} + \omega^{-1} \Delta\vartheta^{(2)}] \, [= \Delta T + \omega^{-1} \Delta\theta] \tag{4.20}$$

the first equation directly from (4.10), the second from (2.16) and (4.18).

The most important consequence of (4.20) is this: for $\Delta t^{(1)} = \Delta t^{(2)} = 0$ (4.20) yields $\kappa_1 \Delta\vartheta^{(1)} = \kappa_2 \Delta\vartheta^{(2)}$ and hence, in view of (4.18), $\kappa_1^2 = \kappa_2^2$, i.e., $r_1 = r_2$. Thus:

$$\boxed{\Delta t^{(1)} = \Delta t^{(2)} = 0 \qquad \text{implies } r_1 = r_2} \tag{4.21}$$

Since

$$\Delta t^{(r)} = 0 \qquad \text{for all values of } r$$

is the defining condition for an *'intrinsic geometry on the rotating disk'*, it follows that *no such geometry exists in Special Relativity.* Instead, we merely have the angular relation (4.18). If, on the other hand, we were to use as defining condition $\Delta\bar{t} = 0$ with any frame coordinate $\bar{t}$ from the set defined by (4.13), the result would depend on the choice of the arbitrary function f appearing in (4.13); in particular, with any choice leading to $\Delta\bar{t} = \Delta T$ we would obtain, not the intrinsic geometry, but the *relative geometry with respect to* (as 'judged from') *the inertial frame* Σ^0.

Contrary to what is suggested by Einstein's fictitious experiment, namely that the circumference of a rotating disk as measured on the disk remains $2\pi r$, as pointed out in the Introduction, (4.18) gives for the circumference

$$C(r) = \kappa_r 2\pi r \tag{4.22}$$

and thus confirms Einstein's conclusion: a beautiful example of the logical fact that a conclusion may be right even if it is wrongly established. Note that in the theory presented here (4.22) is a consequence of

$$\vartheta^{(r)}_{\max} = \kappa_r \, 2\pi \tag{4.23}$$

and the fundamental assumption, shared with all other authors, that the radius is invariant under the transformation $(r = R)$.

Summarising, we have the following results:(i) The intrinsic geometry on the surface of a rotating cylinder is well-defined and Euclidian, as follows from equation (2.19). (ii) An intrinsic geometry on the surface of a rotating disk does not exist and could be defined only in an arbitrary way. (iii) There exist a sort of truncated intrinsic geometry on the rotating disk, concerning angular measures at different radii, leading to the Einstein prediction $C/D = \kappa_r \pi$.

(f) *Kinematics in the rotating frame*

The kinematics in the rotating frame follows from the standard kinematics in a Minkowski (Inertial) frame by applying the transformation (2.14). As this is a transformation to the subframe $r = R$, the transformed equations will contain the parameter R which, together with the relational parameter $\omega = \omega^0$, serves to identify the subframe.

As an example we consider the propagation of light in the rotating frame. Let the light be emitted from a source at the origin of Σ^0 at $T = 0$. A certain element of the wave front can then be described by the kinematic equations

$$Z = 0, \qquad R = cT, \qquad \theta = 0 \tag{4.24}$$

The transformed equations read

$$z = 0, \qquad r = \kappa_R^{-1}\, ct^{(R)}, \qquad \vartheta^{(R)} = -\kappa_R\, \omega r/c \tag{4.25}$$

Hence,

$$\dot{r} = c\kappa_R^{-1}, \qquad \dot{\vartheta}^{(R)} = -\kappa_R\, \dot{r}\omega/c = -\omega \tag{4.26}$$

so that

$$\dot{r}^2 + r^2\dot{\vartheta}^2 = c^2\,[\kappa_R^{-2} + r^2\omega^2/c^2] = c^2 \tag{4.27}$$

Thus, while a ray of light appears curved according to (4.25), its speed remains c, in contrast to the speed of light in a true gravitational field

(g) *The transformation (2.14) in Carthesian coordinates*

For some purposes it may be useful to write the basic kinematic transformation (2.14) in Carthesian and quasi-Carthesian coordinates, i.e., in the coordinates

$$\begin{aligned} X &= R\cos\theta \qquad & x^{(r)} &=_{df} r\cos\vartheta^{(r)} \\ Y &= R\sin\theta \qquad & y^{(r)} &=_{df} r\sin\vartheta^{(r)} \end{aligned} \tag{4.28}$$

The result can be written in the form

$$\left.\begin{aligned} x^{(r)} &= X\,\frac{\cos\kappa_r\theta}{\cos\theta}\cos\kappa_r\omega T + Y\,\frac{\sin\kappa_r\theta}{\sin\theta}\sin\kappa_r\omega T \\ y^{(r)} &= -X\,\frac{\cos\kappa_r\theta}{\cos\theta}\sin\kappa_r\omega T + Y\,\frac{\sin\kappa_r\theta}{\sin\theta}\cos\kappa_r\omega T \end{aligned}\right\} \tag{4.29}$$

where $\theta = \operatorname{arctg}(Y/X)$.

5. *Concluding Remarks*

The theory presented above is an extension of standard Special Relativity to rotating frames; as such it is not necessarily the only one possible. However, it is the only one satisfying the theoretical constraints and the invariance postulate $r = R$. If the latter is maintained the theoretical constraints are well-established. If it is not maintained, the transformation problem, or rather its irreducible core, is a three-dimensional one, and some of the theoretical constraints would have to be reformulated. If '$r = R$' is replaced by '$r = f(\kappa_r)R'$,† the best one could hope for is that the unknown function f together with the other equations is *uniquely* determined by the theoretical constraints. If this should be the case, it is unlikely that the resulting theory is different from the present one, i.e., we would expect $f(x) \equiv 1$. In the alternative case that (at least) one function remains undetermined one would have to look for further constraints.

The author hopes that the present paper may stimulate work in this direction.

Appendix

Discussion of Møller's method

According to the method used by Møller (1952) the intrinsic geometry in any frame in which $ds^2 = g_{ik}\, dx^\iota dx^\kappa$, is given by

$$d\sigma^2 = \gamma_{\iota\kappa}\, dx^\iota\, dx^\kappa \qquad (\iota, \kappa = 1, 2, 3) \tag{A.1}$$

with

$$\gamma_{\iota\kappa} =_{df} g_{\iota\kappa} + \gamma_\iota \gamma_\kappa, \qquad \gamma_\iota =_{df} \frac{g_{\iota 4}}{\sqrt{(-g_{44})}} \tag{A.2}$$

No restriction is imposed on the coordinates used except that x^4 is the only time-like coordinate. Hence the method allows the use of a Galilei transformation for the introduction of frame coordinates in the 'moving' frame:

$$\begin{aligned} x &= X - vT \\ y &= Y \\ z &= Z \\ t &= T \end{aligned} \tag{A.3}$$

Hence

$$\begin{aligned} ds^2 &= dX^2 + dY^2 + dZ^2 - c^2\, dT^2 \\ &= dx^2 + dy^2 + dz^2 + 2v\, dx\, dt - \kappa^{-2} c^2\, dt^2 \end{aligned} \tag{A.4}$$

With $x^1 = x, x^2 = y, x^3 = z, x^4 = ct$, this gives

$$\gamma_\iota = (\beta\kappa, 0, 0) \tag{A.5}$$

and hence

$$d\sigma^2 = (1 + \beta^2 \kappa^2)\, dx^2 + dy^2 + dz^2 \tag{A.6}$$

Thus, the method gives a non-Euclidian geometry for a frame moving with constant velocity with respect to the original inertial frame.

† *Note added in proof.* This case has meanwhile been investigated by the author. The results, which do not invalidate the present paper, will be published in due time.

Besides giving wrong results, the method presented leads to the following paradox. If the geodesics of the geometry given by (4.6), which are determined by the well-known equations

$$\frac{d}{d\sigma}(\gamma_{\iota\kappa}\dot{x}^{\kappa}) = \frac{1}{2}\frac{\partial\gamma_{\kappa\lambda}}{\partial x^{\iota}}\dot{x}^{\kappa}\dot{x}^{\lambda}, \qquad \dot{x}^{\iota} =_{df} \frac{dx^{\iota}}{d\sigma} \tag{A.7}$$

are worked out, the result is

$$x = a_1 + b_1\sigma, \qquad y = a_2 + b_2\sigma, \qquad z = a_3 + b_3\sigma \tag{A.8}$$

where σ is the parameter on the geodesic. Thus *all geodesics are straight lines in the (x, y, z)-space, although this space has a non-Euclidian metric.*

The mistake involved in the method does not lie in the combined use of (A.3) and (A.4), as it may appear on first sight; indeed, the relativistic invariance of ds^2 is a *weaker* postulate than the separate invariances of $d\sigma^2$ and dt^2 postulated by Galileian kinetmatics. Hence the combined use of a Galilei transformation and the invariance of ds^2 is not in itself contradictory, odd though it is. However, if we stick to combined use of (A.3) and (A.4), consistency requires that a choice be made between (A) Galilei kinematics (Newtonian space-time) and (B) Einstein kinematics (Minkowski space-time). In case (A) we have to put $c = \infty$ and hence $\kappa = 1$ in all final results whereby equation (6) reduces to

$$d\sigma^2 = dx^2 + dy^2 + dz^2 \tag{A.6A}$$

In case (B) we have to complement the Galilei transformation $\mathscr{G}$ by a further (frame-conserving) transformation $\mathscr{T}$ so that the two together, i.e. $\mathscr{T}\mathscr{G}$, equal the Lorentz transformation $\mathscr{L}$: $\mathscr{L} = \mathscr{T}\mathscr{G}$. Now the splitting up of the Lorentz transformation $\mathscr{L}_x$ according to the scheme $\mathscr{L}_x = \mathscr{T}_x\mathscr{G}_x$ is as follows:

$$\left.\begin{aligned} \kappa\begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} &= \begin{pmatrix} \kappa & 0 \\ -\kappa\beta & \kappa^{-1} \end{pmatrix}\begin{pmatrix} 1 & -\beta \\ 0 & 1 \end{pmatrix} \\ \mathscr{L}_x &= \mathscr{F}_x \cdot \mathscr{G}_x \end{aligned}\right\} \tag{A.9}$$

This leads to

$$\underline{\bar{x} = \kappa x}, \qquad \bar{t} \equiv \underline{\bar{t}^{(\bar{x})}} = \kappa^{-1}t - \kappa\beta x = \underline{\kappa^{-1}t - \beta\bar{x}} \tag{A.10}$$

and hence to

$$d\sigma^2 = d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 \qquad (\bar{y} = y, \bar{z} = z) \tag{A.6B}$$

Thus, we do get Euclidean geometry in the uniformly moving frame even if we start with the wrong transformation (A.3), but only after the complementing frame-conserving transformation (A.10) which introduces—from the standpoint of the Galilei transformation (A.3)—*local times* $\bar{t}^{(\bar{x})}$ instead of a common frame time.

As to the geodesics in the uniformly moving frame, it follows from (A.6B) that the latter are straight lines in the variables $\bar{x}, \bar{y}, \bar{z}$. However, as—from the standpoint of the method described—a corresponding time variable $\bar{t}$ common to all points of the frame does not exist, the geodesics in the variables $\bar{x}, \bar{y}, \bar{z}$ cannot be ascribed any geometrical significance in the proper sense of intrinsic geometry. Thus the results of the method described—whether supplemented by the complementing frame-conserving transformation or not—are phony: even if they happen to be right they are wrongly established.

We now turn to rotating frames. Here, the method described will be shown to give *wrong results.*

The splitting up of the correct relativistic transformation $\mathfrak{R}_{r,\omega}$ into a Galilei transformation $\mathfrak{G}_\omega$ and a frame-conserving transformation $\mathfrak{F}_{r,\omega}$ is as follows:

$$\left.\begin{aligned} \kappa_r \begin{pmatrix} 1 & -\omega \\ -\beta_r^2\omega^{-1} & 1 \end{pmatrix} &= \begin{pmatrix} \kappa_r & 0 \\ -\beta_r^2\kappa_r\omega^{-1} & \kappa^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & -\omega \\ 0 & 1 \end{pmatrix} \\ \mathfrak{R}_{r,\omega} &= \mathfrak{F}_{r,\omega} \qquad\qquad \mathfrak{G}_\omega \end{aligned}\right\} \tag{A.11}$$

If $\vartheta(=\theta - \omega T)$ and $t(=T)$ are the coordinates in the rotating frame introduced by $\mathfrak{G}_\omega$, the coordinates obtained from them by the complementing frame-conserving transformation $\mathfrak{F}_{r,\omega}$ are:

$$\vartheta^{(r)} = \kappa_r\vartheta \qquad [=\kappa_r(\theta - T)] \tag{A.12a}$$

$$t^{(r,\vartheta)} = \kappa_r^{-1}t - \kappa_r\beta_r^2\omega^{-1}\vartheta \qquad [=\kappa_r(T - \omega^{-1}\theta)] \tag{A.12b}$$

$$= \kappa_r^{-1}t - \beta_r^2\omega^{-1}\vartheta^{(r)} \tag{A.12b$'$}$$

In these coordinates, which are the correct metrical coordinates from the standpoint of Special Relativity, the line element $d\sigma$ in the plane $z = Z =$ const. is given by

$$d\sigma^2 = dr^2 + r^2\, d\vartheta^{(r)^2} \tag{A.13}$$

as follows from (2.19), Section 2. In the non-time orthogonal system of coordinates used by Møller the line element $d\sigma$ turns out to be given by

$$d\sigma^2 = dr^2 + \kappa_r^2 r^2\, d\vartheta^2 \tag{A.14}$$

in agreement with (A.12a). In spite of this agreement, the geodesics worked out according to (A.7) for the two line elements are entirely different because of the r-depending factor κ_r^2 present in (A.14) but absent in (A.13). The following Table confronts the results. [Note that from the standpoint of Special Relativity $\vartheta^{(r)}$ is the correct metrical coordinate (angular measure) and not an abbreviation for $\kappa_r\vartheta$. In other words, $\vartheta^{(r)}$ is an independent variable that *refers* to, but does not mathematically depend on, r.]

Relativistic Metrical Coordinates $(z = Z = 0)$		Møller's Non-metrical Coordinates $(z = Z = 0)$	
$ds^2 = dr^2 + r^2\, d\vartheta^{(r)^2} - c^2\, dt^{(r)2}$	(A.15)	$ds^2 = dr^2 + r^2\, d\vartheta^2 + 2r^2\omega\, d\vartheta\, dt - \kappa_r^{-2} c^2\, dt^2$	(A.16)

Hence by (2)

$$\left.\begin{aligned} \gamma_\iota &= (0, 0) \\ \gamma_{\iota\iota} &= (1, r^2) \\ \gamma_{\iota\kappa} &= 0 \qquad \text{for } \iota \neq \kappa \end{aligned}\right\} \quad \text{(A.17)} \qquad \left.\begin{aligned} \gamma_c &= (0, \kappa_r r^2 \omega c^{-1}) \\ \gamma_{\iota\iota} &= (1, \kappa_r{}^2 r^2) \\ \gamma_{\iota\kappa} &= 0 \qquad \text{for } \iota \neq \kappa \end{aligned}\right\} \quad \text{(A.18)}$$

Hence by (1)

$$d\sigma^2 = dr^2 + r^2\, d\vartheta^{(r)2} \quad \text{(A.13)} \qquad d\sigma^2 = dr^2 + \kappa_r{}^2 r^2\, d\vartheta^2 \quad \text{(A.14)}$$

Hence (7) reduces for $\iota = 2$ to

$$\frac{d}{d\sigma}(r^2 \dot{\vartheta}^{(r)}) = 0 \qquad \frac{d}{d\sigma}(\kappa_r{}^2 r^2 \dot{\vartheta}) = 0$$

and by integration:

$$\dot{\vartheta}^{(r)} = \alpha r^{-2} \qquad \dot{\vartheta} = \alpha \kappa_r^{-2} r^{-2}$$

while $\gamma_{\iota\kappa}\dot{x}_\iota \dot{x}_\kappa = 1$ yields

$$\dot{r}^2 + r^2 \dot{\vartheta}^{(r)^2} = 1 \qquad \dot{r}^2 + \kappa_r{}^2 r^2 \dot{\vartheta}^2 = 1$$

$$\dot{r} = \pm\sqrt{(1 - \alpha^2 r^{-2}} \qquad \dot{r} = \pm\sqrt{(1 + \alpha^2\omega^2 c^{-2} - \alpha^2 r^{-2})}$$

Hence

$$\frac{dr}{d\vartheta^{(r)}} \equiv \frac{\dot{r}}{\dot{\vartheta}^{(r)}} = \alpha^{-1} r^2 \sqrt{(1 - \alpha^2 r^{-2})} \quad \text{(A.19)} \qquad \frac{\dot{r}}{\dot{\vartheta}} = \kappa_r{}^2 \alpha^{-1} r^2 \sqrt{(1 + \alpha^2\omega^2 c^{-2} - \alpha^2 r^{-2})} \quad \text{(A.20)}$$

and by integration

$$\vartheta^{(r)} = \vartheta_0^{(r)} + \arccos \alpha_r^{-1} \quad \text{(A.21)}$$

The geometry on the rotating disk is Euclidean not only locally [equation (13)] but also globally in so far as the geodesics are straight lines [equation (21)].

The geometry on the rotating disk is non-Euclidean not only globally (equation (20)] but even locally [equation (14)].

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